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A FINITE ELEMENT DISPLACEMENT MODEL VALID FOR ANY VALUE OF THE COMPRESSIBILITY

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Abstract—In this paper it is shown how the displacement formulation of the theorem of minimum potential energy can be used with the finite element method to approximate both compressible and incompressible equilibria of linearly elastic, isotropic solids. The procedure is shown to be equivalent to the more complicated "mixed principle" technique, due to the use of numerical integration applied to the computation of the element stiffness matrices. Criteria for the choice of integration formulas and elements are discussed, and numerical examples are presented.

INTRODUCTION

In the linear theory of elasticity a distinction has traditionally been made between the compressible and incompressible cases, because at the point of exact incompressibility, the displacement form of the compressible field equations contains terms which become infinite. The field equations for the incompressible case form an expanded system with an additional equation and a new unknown, the hydrostatic pressure. L. R. Herrmann and his colleagues [1–3] have shown that the form of the field equations for the incompressible case leads to a displacement-pressure formulation valid for all values of the compressibility, and that there is an associated variational statement of the equilibrium.

The elegance of the new variational statement notwithstanding, it is more computationally cumbersome than the variational statement of equilibrium in the compressible case, the theorem of minimum potential energy, when approximations are sought on finite-degree-of-freedom trial spaces. This is in part due to the introduction of additional pressure unknowns, and in part due to the lack of positive definiteness of the "mixed" Herrmann variational principle. To maximize computational efficiency, computations are often performed using the standard formulation for solutions in the compressible range, and using the mixed formulation for nearly incompressible and incompressible solutions. It will be shown that one unified approach may be taken, and one computer program may be used for all values of the compressibility. It will be shown that the incompressible case can be handled by choosing a small but non-zero value of the compressibility for which the "compressibility error" is on the same level as other sources of error. Numerical integration of the element stiffness matrices is used in such a way as to make the numerically integrated theorem of minimum potential energy mathematically equivalent to the Herrmann variational statement, thus combining the computational advantages of the former with the theoretical desirability of the latter.

THE VARIATIONAL APPROXIMATIONS

The stresses and strains are put into the usual vector form [4],‡ and the tensor of isotropic elastic constants C_{ijkl} has a representation as a 6×6 matrix $\mathbf{D}$ so that stress is related to strain by $\boldsymbol{\sigma} = \mathbf{De}$. $\mathbf{D}$ has the form

$$\mathbf{D} = 2G\{\mathbf{I} + z\mathbf{E}\} \tag{1}$$

where $\mathbf{I}$ is the 6×6 identity, and $E_{ij} = 1$, $i, j \leq 3$ and zero elsewhere. $z \equiv \nu/(1-2\nu)$; G is the shear

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‡It is convenient here not to multiply the shear components by two in contrast to [4].

modulus and ν Poisson's ratio. By using **D** in the derivation of the form of the theorem of minimum potential energy, it follows that the solution **u** to the compressible equilibrium problem minimizes

$$I(\mathbf{u}) = G\int_{\Omega} \{z\theta^2 + e_1^2 + e_2^2 + e_3^2 + 2(e_4^2 + e_5^2 + e_6^2)\}\, dv \tag{2}$$

where θ is the volume strain associated with the displacement field **u**, and the e_i are the components of **e** associated with **u**. Ω is the reference configuration of the solid.

To obtain approximate solutions the finite element method is used. Following the derivation of [4], a displacement trial space S^h is constructed using C^0 conforming elements (appropriate to second order problems) and polynomial shape functions. h is the maximum element diameter. The virtual work argument of [4] can be used to derive the matrix equations. By using **D** in this form in the derivation, it is found that the stiffness matrix has the following form:

$$\mathbf{K} = 2G(\mathbf{K}_2 + z\mathbf{K}_1). \tag{3}$$

It will be assumed that the essential b.c. are such that the Korn inequality [5] guarantees the positive definiteness of the "deviatoric" matrix $\mathbf{K}_2$. The "volumetric" matrix $\mathbf{K}_1$ is semidefinite, and thus the discrete problem unique solution.

For Herrmann's principle the same displacement trial space S^h is used, but another trial space T^h is needed for the hydrostatic pressure H. Conformity is not required as H is not differentiated in the Herrmann functional

$$J(\mathbf{u}, H) = G\int_{\Omega} \{e_1^2 + e_2^2 + e_3^2 + 2(e_4^2 + e_5^2 + e_6^2) + 2\nu\theta H - \nu(1-2\nu)H^2\}\, dv. \tag{4}$$

Shape functions using the same geometrical elements as S^h with a (possibly) different arrangement of nodes are chosen. The full trial space will be denoted by (S^h, T^h). The matrix of the equations of stationarity of J on (S^h, T^h) will be denoted by $\mathbf{K}_H$.

AN ERROR-BALANCING TECHNIQUE

In [6] Fried showed that the energy error introduced by assuming that a linearly elastic, incompressible solid is slightly compressible is $0(\epsilon)$ where $\epsilon \equiv 1 - 2\nu$, and where an artificial $\nu < \frac{1}{2}$ is chosen reflecting the slight compressibility. Additional arguments from [7] lead to

$$\|\mathbf{u} - \mathbf{u}^*\|_1 \leq C_1\epsilon^{1/2} \tag{5a}$$

$$\|H - H^*\|_0 \leq C_2\epsilon \tag{5b}$$

where **u** is the slightly compressible displacement field and **u*** the incompressible field. Let θ be the volume strain associated with **u**; H is the compressible pressure obtained from taking $H = \theta/\epsilon$, using the artificial ν, and H^* is the incompressible pressure. The $C_i > 0$ are constants independent of ν, and the norms are the Sobolev norms of order one and zero. In [6, 7] it is assumed that the shear modulus G remains constant while ν varies, implying the variation of Young's modulus :

$$E = E(\nu) \equiv 2G(1+\nu). \tag{6}$$

In [6] Fried uses (5) on which to base his proposal of "residual energy balancing" as a means of using (2) to approximate incompressible equilibria. Using the displacements as an example, if p is the degree of S^h, (5a) leads to

$$\|\mathbf{u}^h - \mathbf{u}^*\|_1 \leq \|\mathbf{u}^h - \mathbf{u}\|_1 + \|\mathbf{u} + \mathbf{u}^*\|_1 \leq C_3h^p + C_1\epsilon^{1/2} \tag{7}$$

where C_3h^p is the discretization error in approximation of **u**, the exact minimum of (2) with the artificial ν, by the finite element approximation $\mathbf{u}^h$ using the same ν. $C_1\epsilon^{1/2}$ is the error in

approximating the incompressible solution $\mathbf{u}^*$ by $\mathbf{u}$ from (5a). The idea is to choose ν so that $\epsilon^{1/2} = 0(C_3 h^p)$. But unfortunately it is well known that the approximation capabilities of (2) on S^h can deteriorate drastically as $\nu \to \frac{1}{2}$. Close inspection of the error bound of [8] shows $C_3 = C_3'/\epsilon^{1/2}$ which—if it were sharp—would indicate that the best possible combined error could be achieved by choosing $\epsilon = 0(h^p)$ (or $z \equiv \nu/\epsilon = 0(h^{-p})$). This would lead to total error in (7) of $0(h^{p/2})$—or half the power of h in the compressible finite element error bound [8]. Experiments show that this deterioration can occur in practice (see Figs. 1 and 2 and also [1]). Further experiments verify that with $z = 0(h^{-p})$ a slow convergence rate can be achieved (see Fig. 1), but it is a poor rate of convergence.

In order to make the error-balancing work, it will be necessary to modify the theorem of minimum potential energy on S^h in such a way as to drastically reduce the coupling between the value of ν and the discretization error—or better still, to completely uncouple the discretization error from the value of ν, as can be done by proper choice of elements with (4).

In [6] Fried observed that the application of numerical integration apparently achieved the desired uncoupling in an axisymmetric sphere problem. He offered the explanation that $\mathbf{K}_1$ has no null-space, and thus allowing $z \to \infty$ in (3) must force $\mathbf{u} \to 0$. Numerical integration singularizes $\mathbf{K}_1$ so that as $z \to \infty$, the solution is forced into the null-space rather than to zero. However, numerical experiments show accuracy deterioration can occur even if $\mathbf{K}_1$ is singular when exactly evaluated (Fig. 1). In [9] Naylor discusses a series of experiments in which he used numerical integration to evaluate $\mathbf{K}$ and found that by letting $\nu \to \frac{1}{2}$, in spite of wild stress oscillation in each element, accurate stress values could be obtained by sampling at the integration points. He observed that

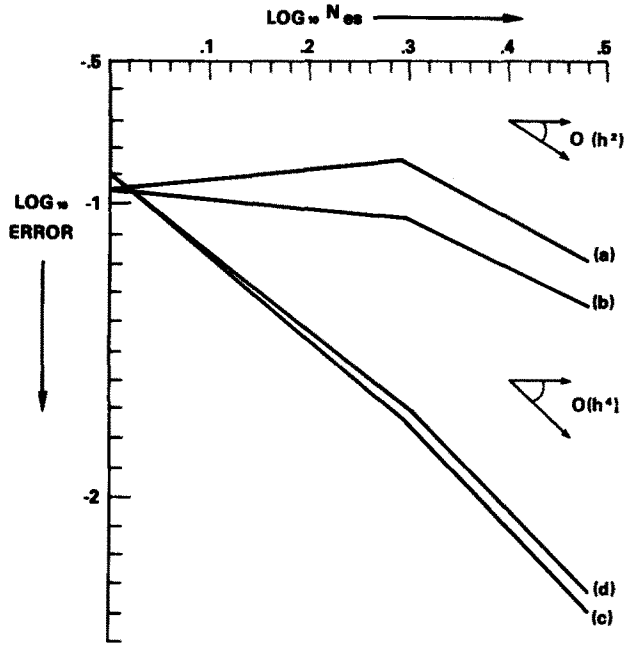


Fig. 1. $\|\mathbf{u}^h - \mathbf{u}^*\|$ with biquadratic Lagrange elements, using different choices of z and integration formulas for $\hat{\mathbf{k}}_{1\epsilon}$: (a) exact integration (3×3 Gauss) and $z = 0(h^{-2p})$, (b) exact integration and $z = 0(h^{-p})$, (c) 2×2 Gauss and $z = 0(h^{-2p})$, (d) the compressible case ($\nu = 0.3$).

the number of integration points used in evaluating $\mathbf{K}$ was crucial in determining whether or not this would work. He made an attempt to predict the success of the numerical integration technique by relating the number of degrees of freedom (henceforth "DOF") in S^h to the number of integration points in the mesh. Independently, this author has performed a number of experiments similar to those in [9] with very similar results. It was found that the basic conclusion that numerical integration enables the uncoupling of discretization error from the value of ν is a sound one, and it does indeed involve the size of the null-space of $\mathbf{K}_1$, which in turn involves the relationship between the number of DOF in S^h and the number of integration points, as well as other factors not yet touched upon.

AN EQUIVALENCE THEOREM

In computational practice it is often inconvenient to exactly evaluate the integrals in the element matrices, so numerical integration is used to evaluate them. $\{w_k^e\}$ will denote the weights and $\{\xi^e = (\xi_1^e, \xi_2^e, \xi_3^e)\}$ the evaluation points of the integration formula used in element e. A " ^ " over any matrix will indicate it has been evaluated by numerical integration. The same or different numerical integration formula could be applied to the evaluation of the deviatoric element matrices $\hat{\mathbf{k}}_{2e}$ as is applied to the volumetric element matrices $\hat{\mathbf{k}}_{1e}$. It will be assumed that $\hat{\mathbf{k}}_{2e}$ are evaluated by an integration formula which satisfies the stability and accuracy conditions of [10, 8]. $\hat{\mathbf{K}}_2$ is then positive definite. The choice of formulas for the $\hat{\mathbf{k}}_{1e}$ will be seen to be governed by other criteria.

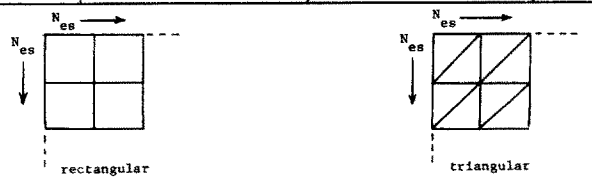
Numerical integration can also be used to evaluate $\hat{\mathbf{K}}_H$. In what follows it will be assumed that the same integration formula is used in each element to evaluate the e_i^2 terms in (4) as is used to evaluate $\hat{\mathbf{k}}_{2e}$. Likewise, it is assumed that the same formula is used in each element to evaluate the terms containing H in (4) as is used for $\hat{\mathbf{k}}_{1e}$. To obtain the main result of this section it is also necessary to assume the following: First $w_k^e > 0$. Second, the $\{\xi_k^e\}$ are chosen so that in the local cöordinate system—before any curving or isoperimetry—a polynomial element shape function basis may be constructed in standard fashion, using the ξ_k^e for nodes—examples are cited in Tables 1 and 2. Third, these element shape function bases are assembled into a global basis of a trial space T^h without enforcing interelement continuity—this can be done even when more than one ξ_k^e is mapped into one ξ_k by assigning a separate nodal value to each occurence of ξ_k as a ξ_k^e in some element e. Such a T^h contains the nodal interpolate even to functions which are discontinuous at interelement boundaries. Under these assumptions, the use of numerical integration serves as more than just a convenient tool for the evaluation of integrals. The following theorem can be proved:

Theorem: The solution $\mathbf{u}^h \in S^h$ obtained by minimizing $I(\mathbf{u})$ on S^h, making use of numerical integration to evaluate the element stiffness matrices, is also the displacement part of a stationary point $(\mathbf{u}^h, H^h)$ of $J(\mathbf{u}, H)$ on (S^h, T^h), where the same integration formulas used for $I(\mathbf{u})$ are used to evaluate the element matrices associated with $J(\mathbf{u}, H)$. The nodes of T^h are the integration points used to evaluate the volumetric terms of $I(\mathbf{u})$ and the terms in $J(\mathbf{u}, H)$ containing H. If θ^h

Table 1. Elements/integration formulas in 2 dimensions passing *R*.II and *R*.III. DOF calculated up to $0(N_{es})$

S^h Element Type	Integration Formula	T^h Element Type	DOF(Z)
Bilinear Lagrange	1 x 1 Gauss	Constant	N_{es}^2
Linear Triangle	1 pt., d.p.* = 1	Constant	$0(N_{es})$
Biquadratic Lagrange	2 x 2 Gauss	Bilinear Lagrange	$4N_{es}^2$
Quadratic Triangle	3 pt., d.p. = 2	Linear Triangle	$2N_{es}^2$
Quadratic Serendipity	2 x 2 Gauss	Bilinear Lagrange	$2N_{es}^2$
Bicubic Lagrange	3 x 3 Gauss	Biquadratic Lagrange	$9N_{es}^2$
Cubic Triangle	6 pt., d.p. = 4	Quadratic Triangle	$6N_{es}^2$
Cubic Serendipity	3 x 3 Gauss	Biquadratic Lagrange	N_{es}^2

Mesh Types:



* "degree of precision"

Table 2. Elements/integration formulas in 3 dimensions passing *R*.II and *R*.III. DOF calculated up to $0(N_{es}^3)$*

S^h Element Type	Integration Formula	T^h Element Type	DOF(Z)
Trilinear Lagrange	1 x 1 x 1 Gauss	Constant	$2N_{es}^3$
Linear Tetrahedron	1 pt., d.p.† = 1	Constant	0
Triquadratic Lagrange	2 x 2 x 2 Gauss	Trilinear Lagrange	$16N_{es}^3$
Quadratic Tetrahedron	4 pt., d.p. = 2	Linear Tetrahedron	N_{es}^3
Quadratic Serendipity	2 x 2 x 2 Gauss	Trilinear Lagrange	$4N_{es}^3$
Tricubic Lagrange	3 x 3 x 3 Gauss	Triquadratic Lagrange	$54N_{es}^3$
Cubic Tetrahedron	10 pt., d.p. = 4	Quadratic Tetrahedron	$19N_{es}^3$
Cubic Serendipity	3 x 3 x 3 Gauss	Triquadratic Lagrange	0

*Mesh types are 3-D generalizations of Table 1 meshes.

†"degree of precision"

is the volume strain associated with $\mathbf{u}^h$ and $\nu < \frac{1}{2}$ is the Poisson's ratio appearing in $I(\mathbf{u})$ and $J(\mathbf{u}, H)$ then $\theta^h/(1-2\nu)$ interpolates to H^h at the integration points.

The proof of this theorem can be found in [7]. For $\nu = \frac{1}{2}$, $J(\mathbf{u}, H)$ becomes a Lagrange multiplier method with H the Lagrange multiplier. Allowing $z \to \infty$ in $I(\mathbf{u})$ makes (2) into a "penalty method"[11] with penalty z enforcing zero volume strain as $z \to \infty$. The theorem presented here may be related to theorems which identify penalty function methods with Lagrange multiplier methods in the limit as the penalty tends to infinity[11]. But this theorem says more, because it associates $I(\mathbf{u})$ with $J(\mathbf{u}, H)$ for each finite z, and identifies trial spaces T^h from which the Lagrange multipliers are drawn.

An important point to note is that the values of θ^h/ϵ at the integration points are the nodal values of H^h, which may be recovered throughout each element by using the shape functions of T^h, and used for stress calculations. Any wild oscillation of θ^h/ϵ between the integration points can be ignored.

The theorem shows that discretization error in (7) can be uncoupled from the value of ν to whatever extent is possible using $J(\mathbf{u}, H)$ on (S^h, T^h). Before we turn our attention to the question of how to choose elements and integration formulas so that (S^h, T^h) produces the desired uncoupling, it is important to note that there is a compelling reason not to take z any larger than necessary to make $C_1\epsilon^{1/2} = 0(h^p)$ in (7). Using arguments in [12], it can be shown that the spectral condition number of $\hat{\mathbf{K}}$, $C_2(\hat{\mathbf{K}})$, is $0(zh^{-2})$. On the other hand, if the discretization error in (7) is on the level of the optimal error bound for the approximation of strains in the finite element method[8], then ϵ can be taken to be $0(h^{2p})$. This leads to an error balance in (7). Taking this into account, it can be seen that to obtain the displacement formulation valid for any value of the compressibility with associated exact Poisson's ratio ν^*, $\hat{\mathbf{K}}$ is used with z chosen according to

$$z = \begin{cases} ch^{-2p}, & \nu = \dfrac{z}{1+2z} < \nu^* \\ \nu^*/(1-2\nu^*), & \text{otherwise.} \end{cases} \tag{8}$$

For compressible materials this amounts to using ν^* always; for incompressible materials an artificial ν is always used. Nearly incompressible materials are treated as incompressible until ν^* is reached. Alternatively, for nearly incompressible materials, $\nu = \nu^*$ could be used for all h, by-passing (8). However, if ν^* is very close to $\frac{1}{2}$, this could adversely affect $C_2(\hat{\mathbf{K}})$, and it might be better to treat the material is incompressible via (8). The choice of "c" in (8) is not crucial, the power of h being the important choice. When the z of (8) is used,

$$C_2(\hat{\mathbf{K}}) = 0(h^{-2p-2}). \tag{9}$$

The increase of the condition number is a price that is paid in the use of this method ($C_2(\hat{\mathbf{K}}) = 0(h^{-2})$ in the compressible case).

CHOICE OF ELEMENTS AND INTEGRATION FORMULAS

The question of achieving the desired uncoupling of errors in the two terms of the right-hand side of (7) can be viewed in terms of constructing an (S^h, T^h), as described in the previous section, in such a way as to achieve this uncoupling for $J(\mathbf{u}, H)$. Inspection of (4) and $\hat{\mathbf{K}}_H$ shows that, on (S^h, T^h), the integration points (nodes of T^h) are points at which the exact incompressibility of $\mathbf{u}^h(\nu)$ is enforced when $\nu = \frac{1}{2}$. Analogously the integration points fulfill the same role in the limit when $z \to \infty$ in (2), and it can be seen that, in the matrix equations associated with (3), allowing $z \to \infty$ forces $\mathbf{u}^h(\nu)$ to have a progressively larger component in the null-space of $\hat{\mathbf{K}}_1$. This null-space will be denoted by Z in what follows. Choosing the element types and integration formulas then can be seen to imply a choice of the number of incompressibility constraints (number of integration points) and the total number of DOF in the displacements. To uncouple discretization error from the value of ν in $J(\mathbf{u}, H)$, a balance must be achieved between the total DOF in the displacements and the number of incompressibility constraints. It is also required that T^h be able to approximate pressures accurately and that the integration formulas applied to the terms containing H in (4) be sufficiently accurate. These requirements come together in three rules for the choice of elements/integration formulas. Let DOF(Z) be the dimension of Z, and consider those meshes for which there is a well defined "number of elements per side", N_{es}, such that $M = sN_{es}^d + 0(N_{es}^{d-1})$ where s is some positive number, M is the number of displacement nodes, and d is the actual number of spatial dimensions used.

R.I: (S^h, T^h) should be such that $\mathrm{DOF}(Z) = 0(N_{es}^d)$.

R.II: If deg S^h† $= p$, T^h should be a trial space, assembled without shared nodes, with deg $T^h = p - 1$, which contains H^h such that $\|H - H^h\|_0 \le Ch^p \|H\|_p$ for every H with $\|H\|_p < \infty$.

R. III: The integration formula applied to $\hat{\mathbf{K}}_1$ should be accurate enough to do the following when applied to $\hat{\mathbf{K}}_H$:

(a) compute $\int_e \phi\psi \, dx_1 \, dx_2 \, dx_3$ exactly, $\phi, \psi \in T^h$,

(b) compute $\int_e \partial u_i^h / \partial x_k \psi \, dx_1 \, dx_2 \, dx_3$ exactly, $\mathbf{u}^h \in S^h$, $\psi \in T^h$,

(c) in addition to (a) and (b), whenever there are terms of degree higher than deg S^h in S^h or deg T^h in T^h, compute $\int_e u_i^h \psi \, dx_1 \, dx_2 \, dx_3$ exactly, if there are $u_i H$ terms in $J(\mathbf{u}, H)$.

R. I is based on the heuristic argument that there should be only as many DOF in T^h as in one vector component of S^h. *R.*II assures that T^h can approximate pressures to the same order of h that S^h can approximate strains, and *R.*III is needed to assure the perturbation in the solution $\mathbf{u}^h$ due to numerical integration does not exceed $0(h^{2p})$. Details can be found in [7]. In smooth curvilinear coordinate systems, *R.*III is exactly as stated; the Jacobian should not be included in the integrals, and partial differentiation—not covariant—should appear in (b). (c) can be seen only to apply in non-cartesian coordinates, because in cartesian coordinates, no $u_i H$ terms occur in $J(\mathbf{u}, H)$.

Tables 1 and 2 illustrate some possible choices of elements and integration formulas which pass *R.*II and *R.*III (for the indicated choice of T^h, which is in general not unique, given a set of integration points). The tables give DOF (Z) in the illustrated two and three dimensional meshes in which two triangles occupy the same area as one rectangle, or five tetrahedra occupy the same volume as one rectangular parallelepiped. The element types/shape functions are as described in [4]. These can be generalized in the construction of T^h, to the case in which all nodes are integration points interior to the element. The integration formulas are found in [4] and [8]. "Gauss" refers to the usual Gauss-Legendre product formulas. The DOF are given to the nearest $0(N_{es}^{d-1})$. The main points illustrated by the tables are that for each degree, DOF(Z) is largest for Lagrange elements; DOF(Z) increases with increasing degree for Lagrange elements, triangles and tetrahedra, but decreases for serendipity elements. One dimensional elements are not shown in tabular form, since it can be shown[7] that, for these elements, in deg $S^h = p$, basing T^h on p

†"Degree of S^h," i.e., the highest degree of a complete polynomial in S^h.

Gauss points always assures that the error uncoupling in (7) can be achieved, even though *R.* I is violated.

NUMERICAL EXPERIMENTS

As a test problem consider the plane-strain model of an infinite strip with a rectangular cross-section, simply supported on opposing sides, loaded on the top surface by a load varying between the supports like $q \cos x$ on $[-\pi/2, \pi/2]$, for some constant q, and free on the bottom surface. The exact solution can be obtained and involves the products of circular and hyperbolic sines and cosines. As a measure of displacement error

$$\|\mathbf{u}^h - \mathbf{u}^*\| \equiv |\mathbf{u}^h - \mathbf{u}^*|_2^2 / |\mathbf{u}^*|_2^2 \tag{10}$$

is used, where $|\cdot|_2^2$ is the square sum of nodal values, $\mathbf{u}^h$ is the approximate solution and $\mathbf{u}^*$ the exact. Meshes like those illustrated on Table 1 were used.

Figure 1 shows that with biquadratic Lagrange elements, the accuracy in $\|\cdot\|$ is poor with $z = 0(h^{-2p})$ and exact integration. Only slight improvement is obtained by taking $z = 0(h^{-p})$, but when the 2×2 Gauss formula is used, the accuracy in $\|\cdot\|$ compares very favorably to that of the compressible case ($\nu = 0.3$). The 2×2 Gauss formula passes all three rules of the previous section. Figure 2 shows that poor results can be obtained when *R.*I is violated while the other two rules are passed. Figure 3 shows $\|\mathbf{u}^h - \mathbf{u}\|$ where $\mathbf{u}$ is the exact solution for the artificial ν as in (7). These curves give an indication of the success in uncoupling discretization error from the value of ν for a fixed mesh, as $\nu \to \frac{1}{2}$ ($\nu = 0.4999999$ is the largest ν actually used).

In a variety of similar experiments in various coordinate systems, the same pattern emerges: Lagrange elements provide accuracy in the incompressible case comparable to that of the compressible case, when the integration formulas suggested in the tables are used; violations of any of three rules can lead to loss of accuracy. Using either of the other two element/integration formula families listed in the tables, which pass all rules but have smaller DOF(Z) than the Lagrange elements, can lead to accuracy loss as $\nu \to \frac{1}{2}$. In addition, the following points emerge: in rectangular elements, θ/ϵ oscillates wildly in each element for small ϵ. With Lagrange elements integrated as in Table 1, accurate pressures can be obtained by sampling at the integration points, whereas with serendipity elements, the pressures sampled at the integration points can be markedly less accurate than with Lagrange elements. With triangular elements, the pressures do

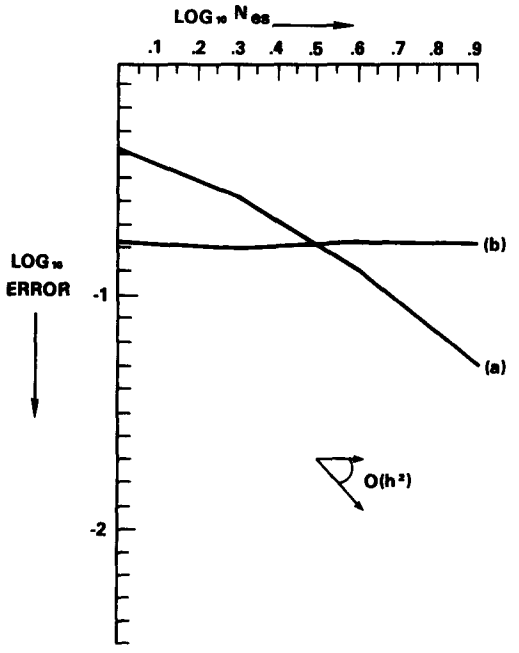


Fig. 2. $\|\mathbf{u}^h - \mathbf{u}^*\|$ with linear triangles and one pt. integration formula for $\hat{\mathbf{k}}_{1\epsilon}$: (a) $z = 0(h^{-2p})$, (b) the compressible case ($\nu = 0.3$).

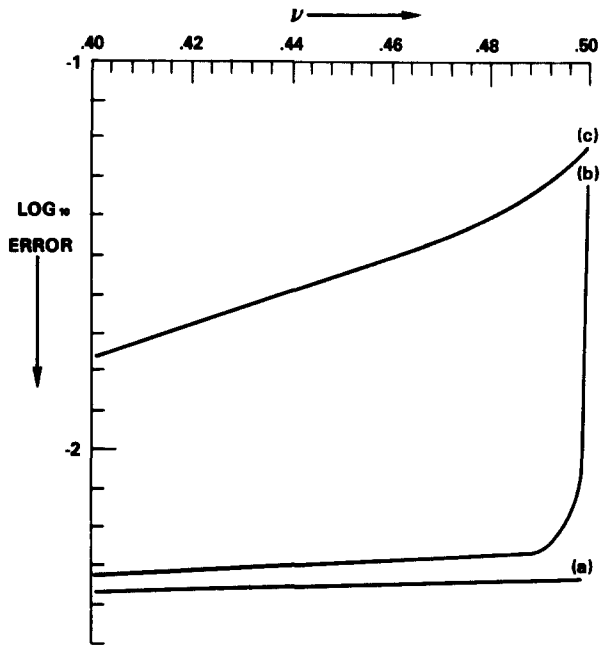


Fig. 3. $\|u^h - u\|$—error in approximating the exact solution with the artificial ν as this $\nu \to \frac{1}{2}$, for three types of quadratic elements, integrated with the formulas of Table 1: (a) Lagrange, (b) serendipity, (c) triangle.

not oscillate wildly but can be much less accurate at the integration points than pressures at the integration points in Lagrange elements. It has also been found that, corresponding to an increase of DOF(Z) with degree, higher degree triangles produce more accurate results than lower degree triangles. In contrast, the accuracy of serendipity elements can decrease with degree to the point of divergence in $\|\cdot\|$ with cubics. The serendipity elements have DOF(Z) which decreases with degree. The numerical experiments so far performed do not establish whether practical use may be made of higher degree triangles and tetrahedra or quadratic serendipity elements. Naylor [9] obtains good results in some problems with quadratic serendipity elements, but in other problems, he reports that he needs to use averaging techniques to obtain accurate results. In numerical experiments performed by this author with Lagrange elements, it has never been necessary to use averaging to obtain accuracy in displacements, strains or pressures comparable to the accuracy obtained in the compressible case.

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